Scale-dependent ocean wave turbulence

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I. Introduction

Wave turbulence is a common feature of nonlinear wave motions observed when external forcing acts during along period of time, resulting in developed spectral cascades of energy, momentum and, possibly, other conserved integrals. In the ocean, wave turbulence occurs on scales from capillary ripples and to those of baroclinic inertia-gravity and Rossby waves.

In general, oceanic wave motions are characterized by rather complicated dispersion laws containing characteristic scales, for instance the Rossby radius of deformation. The resultant absence of scale invariance makes many problems of wave turbulence intractable by standard, small-perturbation-based techniques. As a result, present theoretical understanding has been limited to short- and long-wave asymptotic regimes (Zakharov et al., 1992). Another, more fundamental limitation of the small perturbation theories is the assumption that the wave amplitude be small in relation to the wavelength. Thus, rare (and highly intermittent) events of strongly nonlinear wavelets are disregarded at the outset.

A number of laboratory and field measurements reveal rather peculiar wave spectra which cannot be explained by scale-invariant and/or weak-turbulence theories. The peculiarities include multiple breaks of power laws and saturation of the otherwise monotonous dependence of wave spectra on external forcing (Jähne and Riemer, 1990; Hwartg et al, 1993; Hara et al., 1994; LeTraon et al., 1990). Furthermore, field observations show occurrence of breaking waves in which the nonlinearity is locally very high. In the case of wind-generated surface gravity waves, these are observed as whitecaps. Observation of breaking events in baroclinic inertia-gravity waves requires measurements at a depth of the ocean thermocline - hundred meters below the surface. Breaking waves are manifested as spots of small-scale turbulence resulting from overturning of internal waves at the interface between two layers of different densities. These rare events may coexist quite nicely with the generally low energy level in the wave field. However, their effect on the overall (i.e., averaged over a large time and area) spectrum can be rather important.

These features find a remarkably simple explanation in the framework of a recently developed heuristic approach called “rnultiwave-interaction theory” (Glazman, 1992-95). In what follows we review the basic ideas and results, compare theoretical predictions with experimental data for cases of capillary-gravity and baroclinic inertia-gravity waves and discuss possible avenues for further development of the theory.

II. Observations of ocean wave turbulence

Until recently, the standard example of ocean wave turbulence has been that of deep-water surface gravity waves generated by wind. Characteristic wavelengths of these waves range from lm to 200 m. Earlier field observations, conducted mostly in near-shore regions, showed the "equilibrium" range of the power spectrum as dominated by

$$S(\omega) = \beta g^2 \omega^{-5}, \tag{1.1}$$

where β is a constant. This is known as the Phillips' saturated spectrum (Phillips, 1958). The corresponding 2-d wavenumber spectrum is

$$F(k) = Bk^{-4} \tag{1.2}$$

where $B \approx \beta / 2$. These spectra represent the regime of strong wave turbulence: the equilibrium is reached due to the breaking of steep wave crests, hence due to a highly non-local energy transfer to small scales. The wavelength of breaking waves is within the range described by (1.1)-(1.2). These spectra used to be regarded as universal (Pierson and Moskovitz, 1964; Pierson, 1991). However, later observations - at longer wind fetches - revealed an extended range of frequencies dominated by

$$S(\omega) = \alpha g U \omega^{-4} \tag{1.3}$$

where U is the wind speed (at 10 m height) and α is a constant (Toba, 1973; Forristall, 1981; Kahma, 1981; DoneIan et al., 1985). This range cm-responds to the direct cascade of energy through the spectrum. However, the cascade is not necessarily conservative (Phillips, 1985). In the wavenumber space, spectrum (1.3) takes the form

$$F(k) = Ag^{-1/2} U k^{-7/2} \tag{1.4}$$

where A $\approx \alpha/2$. In terms of the energy flux, Q, through the spectrum, (1.4) can be written as $F(k) = A' g^{2} Q^{1/3} k^{-7/2}$ where A' is the "Kolmogorov constant." Thus, prediction Of weak turbulence theory (Zakharov and Filonenko, 1966) is confirmed.

Zakharov and Zaslavskii (1982) pointed to a possibility of an even flatter spectrum based on the conservation of wave action, P, in an inverse spectral cascade:

$$F(k) = A_p' P^{1/3} k^{-10/3} \quad . \tag{1.5}$$

This spectrum occurs at yet lower frequencies - below the "generation range." An experimental observation of this spectrum by Grose et al. (1972) never received much attention in the oceanographic literature.

A detailed analysis of the gravity wave spectrum for a broad range of wave development stages is presented in Glazman (1994) based *on* buoy observations. In particular, it shows that the exponent p in I?(k)-k-p slowly grows as the wavenumber increases away from the spectral peak. Therefore, different regimes of energy and action flow dominate in different subranges of the spectrum.

Another example of wave turbulence is given by capillary-gravity (CG) waves (Jähne and Riemer, 1990; H wang, et al., 1993; Hara, et al., 1994). "he CG spectra exhibit pronounced "breaks" at certain scales pointing to an important role played by the intrinsic scale $(\sigma/g)^{1/2}$ of the problem. Here σ is the surface tension coefficient divided by the water density and g is the acceleration due to gravity. The CG wave spectra measured at different wind speeds by Hwang et al. (1993) are illustrated in Fig. l(b).

Manifestation of wave turbulence in *long* baroclinic waves (called inertia-gravity (IG) waves) was recently discovered by re-interpreting 1 d wavenumber spectra of sea surface height (SSH) spatial variations on scales 10 to 1000 km (Glazman, 1995(b)). Examples of l-d SSH spectra based on satellite altimeter measurements are reported by Gordon and Baker (1980), Fu (1983), Gaspar and Wunsch (1989), Le Traon et al., (1990; 1994) and others. A typical spectrum is illustrated in Fig. 2. These spectra are dramatically different from what one would observe if SSH variations were dominated by the 2-dimensional eddy turbulence. Indeed, according to Kraichnan's (1967) prediction, the kinetic energy spectrum of 2-d turbulence is given by k^{-3} ox $k^{-5/3}$ laws for the energy and enstrophy cascades, respectively, Assuming geostrophy, these laws translate respectively into k^{-5} or $k^{-11/3}$ spectra for SSH oscillations along altimeter ground tracks. The altimeter-observed SSH spectra are much flatter. Besides, they exhibit spectral breaks pointing to an important role played by the intrinsic spatial scale, the Rossby radius of deformation, characterizing IG wave turbulence.

III. Kinetic equations of weak-turbulence theory

Assuming wave fields to be near Gaussian, a closed form equation for second statistical moments can be derived. by means of small-perturbation techniques (Zakharov et al., 1992). For a decay dispersion law - such that 3-wave interactions are resonant - the kinetic equation is

$$\frac{\partial N(\mathbf{k},t)}{\partial t} = \pi\int [|V_{k12}|^2 f_{k12}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\omega_k-\omega_1-\omega_2) + 2|V_{1k2}|^2 f_{1k2}\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)\delta(\omega_1-\omega_k-\omega_2)]d\mathbf{k}_1 d\mathbf{k}_2 + \gamma(k)N(\mathbf{k},t) \qquad (2.1)$$

where $N(\mathbf{k},t) = F(\mathbf{k},t)/\omega(k)$ is the spectral density of wave action, $F(\mathbf{k},t)$ is the spectral density of wave energy. V_{k12} is the interaction coefficient for wave triads, $\gamma(k)$ is the growth (decay) rate due to external forcing (dissipation). Molecular viscosity corresponds to $\gamma(k) = -2\nu k^2$. A conservative cascade occurs when all external sources and sinks are outside the inertial range, i.e. $\gamma(k) \equiv 0$. Furthermore,

$$f_{k12} = N_1N_2 - N_k(N_1 + N_2),\ N_l = N(\mathbf{k}_l,t) \qquad (2.2)$$

For a non-decay dispersion law - when 3-wave intro-actions are non-resonant - the kinetic equation (after eliminating non-resonant terms by an appropriate canonical transformation) has the form (Zakharov et al., 1992):

$$\frac{\partial N(\mathbf{k},t)}{\partial t} = \frac{\pi}{2}\int [|T_{k123}|^2 f_{k123}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_k+\omega_1-\omega_2-\omega_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 + \gamma(k)N(\mathbf{k},t) \qquad (2.3)$$

where

$$f_{k123} = N_2N_3(N_1+N_k) - N_1N_k(N_2+N_3) \qquad (2.4)$$

and T_{k123} is the interaction coefficient for resonant tetrads. Explicit expressions for V_{k12} and T_{k123} for various wave processes are given, for instance, by Zakharov (1984) and Zakharov et al. (1992). Taking into account the physical significance of the collision integrals in (2.1) and (2.3), these equations can be written in a more instructive form (Phillips, 1977):

$$\frac{\partial N(\mathbf{k},t)}{\partial t} + \nabla_k \cdot \mathbf{P}^{(n)} = \gamma(k) N(\mathbf{k},t) \tag{2.5}$$

where $\nabla_k \cdot$ P(") is the divergence in the wave-vector space of the action flux due to n-wave interactions. In the next section this form is generalized to account for a higher number of resonantly *interacting* Fourier components.

In the simplest, scale-invariant case, the dispersion law and the interaction coefficients are homogeneous functions of their arguments:

$$V(\lambda\mathbf{k},\lambda\mathbf{k}_1,\lambda\mathbf{k}_2) = \lambda^m V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) \qquad T(\lambda\mathbf{k},\lambda\mathbf{k}_1,\lambda\mathbf{k}_2,\lambda\mathbf{k}_3) = \lambda^m T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \tag{2.6}$$

$$\omega(k) = ak^s \tag{2.7}$$

It can be also shown that, regardless of the nature of the wave process, m and s are related by

$$m = 3s/2 \tag{2.8}$$

Assuming the nonlinear interactions to be local in the wavenumber space, equations (2.1), **(2.3), (2.6)** and **(2.7)** yield scaling relationships for the characteristic interaction time (the "turnover time"):

$$t^{-1} \approx N k^{2m} \quad \text{for 3-wave interactions} \tag{2.9a}$$

$$t^{-1} \approx N^2 k^{2m} \quad \text{for 4-wave interactions} \tag{2.9b}$$

The relevant small parameter in the perturbation expansion yielding eqs (2.1) and (2.3) is $\varepsilon = ak$ where a is the characteristic wave amplitude related to $N(k)$ by $N d\mathbf{k} \propto a^2/\omega$. It is easy to check that equations (2.6)-(2.8) allow expressing the turnover time in terms of ε:

$$t^{-1} \approx \omega\varepsilon^2 \tag{2.10a}$$

$$t^{-1} \approx \omega\varepsilon^4 \tag{2.10b}$$

This form remains valid for a broad class of nonlinear wave systems which are not necessarily scale-invariant and whose degree of nonlinearity is measured by a more general quantity

$$\varepsilon = u/c \ , \tag{2.11}$$

where u and c are characteristic particle and phase velocities at scale k, respectively. Only in a case of deep-water waves does (2.11) reduce to $\varepsilon = ak$.

The 1-d form of (2.5) is obtained either by transforming to the frequency space (Zakharov, 1984) or by multiplying by k and integrating over the polar angle in the 2-d wave-vector space. This yields

$$\partial \overline{N} / \partial t + \partial P / \partial k = \gamma \overline{N} \tag{2.12}$$

where $$\overline{N}(k,t) = \int_{-\pi}^{\pi} N(k,\theta,t) k d\theta \quad \text{and} \quad P(k,t) = \hat{i} \int_{-\pi}^{\pi} \mathbf{P}(k,\theta,t) k d\theta$$

Similar expressions can be derived for the wave energy and momentum fluxes. If $\gamma(k) = 0$, the fluxes are conserved in the spectral cascade (Zakharov, 1984; Zakharov et al., 1992). The steady-state inertial cascades of wave action and energy are given by

$$P(k) = P_0 \quad \text{and} \quad Q(k) = Q_0 \tag{2.13}$$

where $Q(k) = \omega(k)P(k)$ is the energy flux.

Weak-turbulence theory is most useful for scale-invariant systems and purely inertial cascades - when solutions of (2.13) are given by power law functions. Departures from (2.6) and (2.7) make the task of solving the collision integral quite formidable. The quasi-Gaussian assumption underling the kinetic equation, even for the lowest degree of wave nonlinearity, disregards possible intermittence in the wave field. Moreover, the small-perturbation approach does not allow one to explore effects of higher-order wave-wave interactions

$$\omega_0 \pm \omega_1 \pm \ldots \pm \omega_m = 0 \quad and \quad \mathbf{k}_0 \pm \mathbf{k}_1 \pm \ldots \pm \mathbf{k}_m = 0 \ , \tag{2.14}$$

in which $m > 4$. These interactions become highly important at least in some localized regions of the wave field, and they may lead to wave tn-caking. Duc to extreme mathematical difficulties of accounting for scale-dependence and high-order nonlinearity, a heuristic approach may have considerable advantages over the formal perturbation theory.

IV. Multiwave interaction approach

The present approach (Glazman, 1992,1993, 1995) is based on a set of scaling relationships which become especially obvious if introduced in the language of the previous section.

Let us consider an inertial energy cascade. The *external* source is assumed to act at lower frequencies - outside our inertial spectral subrange. The rate Q_0 of energy input from an external source, assumed to be known, equals the rate of energy transfer down the spectrum, hence the dissipation rate at high wavenumbers. Following the standard scaling procedure (e.g., Frisch et al., 1978; Larraza et al. 1990), Q is related to the characteristic time of nonlinear interactions between Fourier components, t_j, and to the energy, E', transferred from a cascade step j to step $(j+1)$:

$$Q = E_j / t_j \tag{3.1}$$

To be more specific, one may introduce characteristic wavenumber scales k_j and k_{j+1} marking the (tentative) boundaries of individual cascade steps. The net transfer of energy from longer waves (with characteristic wavelength $2\pi/k_j$) to shorter waves (with characteristic wavelength $2\pi/k_{j+1}$) is similar to the production of smaller eddies by unstable large eddies in the 3-d turbulence. Following this analogy, one can introduce a constant ratio, r, for the cascade process:

$$k_{j+1} = rk_j \tag{3.2}$$

where r > 1. A specific value of r is not required for our subsequent development. Apparently, the energy transferred during time t_j - in a single step of the cascade - is related to the spectral density, F(k), of the wave energy by

$$E_j = \int_{k_j}^{k_{j+1}} \int_{-\pi}^{\pi} G(k,\theta) k d\theta dk = \int_{k_j}^{k_{j+1}} F(k) k dk \tag{3.3}$$

where $G(k,\Theta)$ is the 2-d "angular" spectral density and F(k) is the 2.-d energy spectrum after integrating over all wave propagation directions Θ. The limited width $(k_{j+1} - k_j)$ of a cascade step allows one to introduce characteristic scales for all dynamical quantities at each step j.

Provided t_j can be expressed as

$$t_j = t_j(k_j, E_j) \quad (3.4)$$

equation (3. 1) serves in place of the kinetic equation (2.13) to determine the energy at each step of the cascade:

$$E_j = E_j(Q, k_j) \quad (3.5)$$

The continuous spectrum, F(k), is then found by differentiating (3.3) over k_j and using (3.2): $\partial E_j / k_j = F(k_j r) k_j r^2 - F(k_j) k_j$. If the spectrum falls off sufficiently fast with an increasing wavenumber, the first term in the r.h.s. becomes negligible compared to the second term, and one can write:

$$F(k_j) \approx -\frac{1}{k_j}\frac{\partial E_j}{\partial k_j} \quad (3.6)$$

For a special case of $F(k) \propto k^{-p}$, this approximation is valid if

$$r^{-p+2} << 1 \quad (3.7)$$

Eq. (3.7) replaces a more rigorous criterion (derived in weak turbulence theory) for the wave-wave interactions to be local (e.g., Zakharov et al., 1992).

In a weakly nonlinear case, the turnover time can be formally obtained by scaling the collision integral, (2.9). However, we shall introduce this timescale in a less formal fashion. To this end let us notice that the nonlinearity of most wave processes is measured by *(2.11)* where $c=\omega/k$. Respectively, lowest-order nonlinear terms in deterministic equations of motion (for properly normalized and partially-time-averaged Fourier amplitudes $a(\mathbf{k},t) \propto \varepsilon$) are of order ε^2 and the subsequent terms are of order&3, ε^4, etc. However, since the kinetic equation is derived for second statistical moments (i.e., for the wave action spectral density, $N(k) = F(k)/\omega$), the equations of statistical theory are developed in powers of ε^2: each additional Fourier component accounted for in the interaction integral adds terms ε^2 times as great as a preceding term. Suppose we could derive a general, closed-form equation for $N(k)$ for an arbitrary number of resonantly interacting wave components. Symbolically, this equation can be written as

$$\frac{\partial N(\mathbf{k};t)}{\partial t} + \nabla_k \cdot \mathbf{P}^{(3)} + \nabla_k \cdot \mathbf{P}^{(4)} + \nabla_k \cdot \mathbf{P}^{(5)} + \ldots = \gamma(\mathbf{k}) N(\mathbf{k},t) \tag{3.8}$$

where "partial" collision integrals $\nabla_k \cdot \mathbf{P}^{(n)}$ account for n-wave interactions (of which resonant interactions are most important). $\gamma(k)N(\mathbf{k}, t)$ represents the external source (or sink, or both) where $\gamma(k)$ is an increment (decrement) of wave growth (attenuation). The collision integrals scale as $\varepsilon^{2(n-1)}$. Weakly nonlinear waves (i.e., $\varepsilon << 1$) permit neglecting all collision integrals except for the first one. Indeed, if ε = 0.1, the first interaction term in (3.8) is 10^2 times as large as the subsequent terms. However, this is not the case if the nonlinearity is stronger. For a weak inequality $\varepsilon < 1$, we would have to retain a series of terms (up to n ≈ 6 for the case of $\varepsilon \approx 0.5$) in order to maintain the same accuracy as in our example with $\varepsilon \approx 0.1$. The number of "effective" terms to be retained is thus a function of the degree of the wave nonlinearity, ε. When $\varepsilon \to 1$, interactions of all orders become of comparable importance. This case of strong wave turbulence (i.e., $n \to \infty$) results in "saturated" spectra. Larraza et al. (1990) showed that the Phillips spectrum $F(k) \sim k^{-4}$ for deep-water gravity waves is just one example.

The appropriate characteristic time of resonant wave-wave interactions should be taken as the slowest among all individual turnover times associated with partial fluxes. This corresponds to the highest value of ν among all "effective" collision integrals. Therefore, the appropriate turnover time, found by scaling the terms in (3.8), is

$$t_j^{-1} \approx \omega \varepsilon^{2(\nu-2)} \tag{3.9}$$

Expressing ε in terms of the relevant parameters, namely k, ω and wave amplitude, a(k), or energy Ej at a given step of the cascade, equations (3.4)-(3.6) yield the spectrum of the wave energy. While it is intuitively clear that v should bean increasing function of the external energy input, its determination is an open issue. Some empirical and semi-empirical models have been proposed(Glazman, 1992, 1993, 1995(a)), and one of them is employed in the next section.

IV. 1. Capillary-gravity wave turbulence

We will first consider capillary-gravity waves on deep water for which the ratio of water particle to wave phase velocity is

$$\varepsilon = ak \tag{4.1}$$

The equi-partition between the kinetic and potential energies allows one to express the total wave energy as twice the potential energy:

$$\mathcal{E} = 2\mathcal{E}_p = \rho g \int \eta^2 dx + 2\rho\sigma \int \left(\sqrt{1+|\nabla\eta|^2} - 1\right) dx \tag{4,2}$$

Here, $\eta=\eta(x,t)$ is the elevation of the fluid surface above the zero-mean level, g is the acceleration due to gravity, and σ is the coefficient of surface tension divided by fluid density p. Using the ensemble-averaged form of (4.2) we note that

$$< \eta^2 > = \int_0^\infty F_\eta(k) k dk \quad \text{and} \quad < (\nabla\eta^2) > = \int_0^\infty k^2 F_\eta(k) k dk$$

where $F_\eta(k)$ is the (2-dimensional) spectral density of surface height oscillations averaged over the polar angle Θ. Assuming $|\nabla\eta|^2 << 1$ (which is well justified under natural sea conditions), the surface density of the wave energy is given by

$$E = \rho g \int_0^\infty F_\eta(k)\left[1 + \frac{\sigma}{g}k^2\right] k dk \tag{4.3}$$

Obviously, the spectra of wave energy and surface height are related by

$$F(k) = \rho g F_\eta(k)\left[1 + \frac{\sigma}{g}k^2\right] \tag{4.4}$$

Replacing the semi-infinite integration range by a narrow spectral window one can express Ej in terms of the characteristic wave amplitude, a_j, for a given step in the cascade:

$$E_j = \rho g \int_{k_j}^{k_{j+1}} F_\eta(k)\left[1 + \frac{\sigma}{g}k^2\right] k dk \approx \rho g a_j^2\left(1 + \frac{\sigma}{g}k_j^2\right) \tag{4.5}$$

This equation immediately yields the necessary expression for $E = \varepsilon(E_j, k_j)$ to be used in *(3.9)*. The dispersion law for CG waves is

$$\omega^2 = gk + \sigma k^3 \tag{4.6}$$

Let us introduce the non-dimensional wavenumber

$$K = kL \tag{4.7}$$

where the intrinsic scale of the problem is

$$L = (\sigma/g)^{1/2} \tag{4.8}$$

After a little algebra, eqs. (3.4)-(3.6) and (4.4) yield the power spectrum of surface height spatial variations:

$$F_\eta(k) = B \cdot K^{-(4\nu-11/2)/(\nu-1)}\left(1+K^2\right)^{-3/2(\nu-1)} \cdot \Lambda(K,\nu) \tag{4.9}$$

where

$$B = \alpha'(Q/\rho w^3)^{1/(\nu-1)} L^4 \;, \quad c\%' = \alpha\frac{(2\nu-7/2)}{(\nu-1)} \;, \qquad w = (\sigma g)^{1/4} \tag{4.10a}$$

and

$$\Lambda(K,\nu) = 1 \frac{2\nu-5}{2\nu-7/2} \frac{K^2}{1+K^2} \tag{4.10b}$$

α is a non-dimensional "Kolmogorov constant" of proportionality. In the limits of short and long waves, this spectrum yields Zakharov-Filonenko (1966,1967) power laws for capillary and gravity waves, respective] y.

Using additional expressions for Q and v as functions of wind speed, equations (4.9)-(4.10) describe the (rather complicated) shape of the CG wave spectrum and its dependence on external factors. Beside wind forcing, these factors include the magnitude of the spectrum at the lower-wavenumber boundary of the inertial subrange. By comparing (4.9)-(4. 10) with experimental data, we find that a fixed value of v, such as v = 3 or v = 4, leads to a drastic disagreement with observations (Glazman, 1995(a)). If, however, this effective number is allowed to increase with an increasing wind, the agreement becomes quite reasonable. Figure 1(a) illustrates the predicted spectrum in terms of the "saturation function" $B(k) = k^4 F_\eta(k)$. We used $v = c_0 + c_1 U$ where c_0 and c_1 are empirical coefficients found earlier (Glazman, 1995(a)).

IV.2. Inertia-gravity (IG) wave turbulence

Nonlinear shallow-water equations for a rotating fluid have the form:

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right)\mathbf{U} + f\mathbf{k} \times \mathbf{U} = -g\nabla\eta$$
$$\frac{\partial\eta}{\partial t} + \nabla \cdot ((H+\eta)\mathbf{U}) = 0 \quad (4.11)$$

Here, U is the horizontal velocity vector averaged over the layer depth H, and **k is the** unit vector along the Earth rotation axis. For simplicity, the Coriolis parameter is assumed to be constant (f-plane approximation) and the gravity force g parallel to **k.** (In the rest of this paper, symbol k is employed for a different purpose: it designates the wavenumber vector. We hope this will not cause any confusion.) The nonlinear terms in (4. 11) become especially important in the case of baroclinic waves. Therefore, we shall treat H as the thermocline depth and g as the reduced gravity - implying a 1.5 layer model in which the density of the upper layer is slightly lower than that of the (semi-infinite) lower layer. The amplitude of the density interface oscillation, $\eta(\mathbf{x},t)$, may constitute an appreciable fraction of the thermocline depth. The ocean surface plays only a passive role: its response to the oscillations of $\eta(\mathbf{x},t)$ is very weak and linear and with an opposite sign (e.g., Le Blond and Mysak, 1978). However, since the statistics of the density interface oscillations are identical (up to a constant of proportionality) to those. of the SSH variations, we shall view the spectrum of $\eta(\mathbf{x},t)$ as the SSH spectrum.

The dispersion relationship of the corresponding linear theory is

$$\omega^2 = f^2 + C_0^2 k^2 \text{ where } Co = \sqrt{gH} . \quad (4.12)$$

This equation forbids 3-wave resonance. Therefore, the lowest order resonance occurs in wave tetrads (Falkovich and Medvedev, 1992). The intrinsic scale of the problem (the Rossby radius of deformation) is:

$$R = Co/f \quad (4.13)$$

At high wavenumbers, equations (4,11) describe non-dispersive waves, while the low-wavenumber limit corresponds to inertial ("gyroscopic") waves. The kinetic and potential energies of IG waves (per unit volume, per unit mass of water) are

$$EK = \frac{\langle |U|^2 \rangle}{2}, \quad EP = \frac{g\langle \eta^2 \rangle}{2H}, \tag{4.14}$$

For a narrow frequency band between k_j and k_{j+1}, these energies can be related to the characteristic scales of the wave amplitude, a_j, and wave number, k_j. The energy ratio increases with an increasing wavelength (e.g., [Gill, 1982]). The linearized theory yields

$$\frac{EK_j}{EP_j} \approx 1 + \frac{2}{(kR)^2} \tag{4.15}$$

Physically, the absence of energy equi-partition is due to the fact that the orbits of water particles are not strictly vertical (as would be the case for pure gravity waves). Their inclination is the greater, the larger the relative importance of the Coriolis force. Since the total energy is E=EK+EP, it is useful to express both components in terms of E_j:

$$EK_j \approx \frac{E_j}{2}\left(1 + \frac{1}{1+(kR)^2}\right) \tag{4.16}$$

$$EP_j \approx \frac{E_j}{2} \frac{(kR)^2}{1+(kR)^2} \tag{4.17}$$

In view of (4. 16), the characteristic particle velocity at scale k is given by

$$u^2(k) \approx E_j\left(1 + \frac{1}{1+(kR)^2}\right) \tag{4.18}$$

Based on (4.12) the characteristic phase speed is

$$c^2(k) \approx C_0^2\left(1 + \frac{1}{(kR)^2}\right) \tag{4.19}$$

Now we can express the interaction time, t_j, in terms of E_j, k, $\omega(k)$ and C_0:

$$t_j^{-1} \approx C_0 R^{-1}\left(1+K\right)^{1/2}\left[\frac{E_j}{C_0^2}\left(1 - \frac{1}{(1+K^2)^2}\right)\right]^{\nu-2} \tag{4.20}$$

where $$K = kR \tag{4.21}$$

is the non-dimensional wavenumber. It is al so convenient to non-dimensionalize other quantities:

$$\tilde{Q}_0 = Q_0(R/C_0^3), \quad \tilde{E}_j = E_j/C_0^2, \quad \tilde{F}(\tilde{k}) = F(k)/(C_0R)^2 \tag{4.22}$$

With t_j^{-1} given by (4.20), equation (3.1) can be solved for E_j. In the non-dimensional form, the result is

$$\tilde{E}_j \approx \tilde{Q}_0^{1/(\nu-1)} z^{-1/2(\nu-1)} (1 - 1/z^2)^{-(\nu-2)/(\nu-1)} \qquad (4.23)$$

where we introduced

$$z = 1 + K^2 \qquad (4.24)$$

According to (4.12), this variable has a simple physical interpretation: $z=(\omega(k)/f)^2$.

Equation (3.6) becomes

$$\tilde{F}(K) \approx -2\left.\frac{\partial E_j}{\partial z}\right|_{z=1+K^2} \qquad (4.25)$$

This yields the 2-d spectrum of the total wave energy:

$$\tilde{F}(K) = \alpha \frac{\tilde{Q}_0^{1/(\nu-1)}}{(\nu-1)} z^{(\nu-7/2)/(\nu-1)} (z^2-1)^{-(2\nu-3)/(\nu-1)} (z^2+4\nu-9) \qquad (4.26)$$

The ("Kolmogorov") constant, α, enables us to replace sign" =" with" =". The surface height spectrum (i.e., the potential energy spectrum) is found based on (4.17):

$$\tilde{F}_\eta(K) = \alpha \frac{\tilde{Q}_0^{1/(\nu-1)}}{2(\nu-1)} \frac{(z-1)(z^2+4\nu-9)}{(z^2-1)^{(2\nu-3)/(\nu-1)} z^{5/2(\nu-1)}} \qquad (4.27)$$

Since the angular dependence in our 2-d spectra is forgone, the corresponding 1-d spectrum is simply $\tilde{F}_\zeta(K)K$. The plot of this spectrum is shown in Fig. 3 for several values of v. In the high-wavenumber limit, the spectrum behaves as K^{-s} where $s=1+1/(v-1)$. Apparently, the regime of $v \to \infty$ corresponds to a surface which is discontinuous in the mean square. Physically, this means that short waves make bores ("shocks") and break, and the energy cascade becomes non-local. The main oceanographic implication of this internal wave breaking process is the production of small-scale turbulence in the ocean thermocline and an increased vertical mixing. Being relevant for the direct energy cascade, (4.26) applies to wavenumbers above the generation range. At lower wavenumbers, one must consider the inverse cascade.

For weakly non-linear waves (when v=: 4), the kinetic equation admits an additional physically-meaningful solution corresponding to the inverse cascade of wave action

(Falkovich and Medvedev, 1992). Since it is the wave action that is conserved in the cascade, equation (3. 1) is replaced by

$$P_0 = N_j / t_j \tag{4.28}$$

where $N_j = \int_{k_n}^{k_{n+1}} F(k)\omega^{-1} k dk$. In the same fashion as before, we ultimately arrive at:

$$\tilde{F}(K) = \beta \frac{2\tilde{P}_0^{1/3}}{3} z^{1/3}(z^2-1)^{-5/3} \tag{4.29}$$

The spectrum of surface height variations becomes

$$\tilde{F}_\eta(K) = \beta \frac{\tilde{P}_0^{1/3}}{3} z^{-2/3}(z^2-1)^{-5/3}(z-1) \tag{4.30}$$

The "Kolmogorov" constant β is different from that appearing in (4.26). Obviously, these two constants are related because (4.29) must merge with (4.26) at intermediate wavenumbers. Figure 3 shows that the spectra merge quite smoothly (at $K \approx 1.2$) for all values of v. In the longwave limit, (4.30) tends to $k^{-4/3}$ which describes purely inertial oscillations whose potential energy tends to zero. Spectrum (4.30) is confirmed by satellite-altimeter observations (Wunsch and Stammer, 1995) on scales much greater than the Rossby radius of deformation.

Plausible sources of baroclinic wave energy include tidal forcing (when barotropic tides interact with topographic features of ocean basins), fluctuations of wind stress and atmospheric pressure, and various types of instability of ocean currents, eddies, etc. In view of a presently poor quantitative knowledge of these sources, it is difficult to suggest any specific dependence of Q_0 and P_0 on external factors. Comparison of theoretical (4.27),(4.30) and observed spectra, Figs. 2 and 3, points to a rather high degree of nonlinearity typical for "short" (KR >> 1) baroclinic waves.

V. Discussion

The heuristic approach described in section 1 V enables one to analyze rather realistic wave processes occurring in oceans, atmosphere, and other nonlinear media. At the present time, the most urgent task is to gain a quantitative understanding of the effective number **of** interacting wave harmonics, v, as a function of the external factors.

The present approach can be applied to other cases of wave turbulence - such as Rossby waves, deep-water internal waves, etc. The fact that spectra of scale-dependent wave turbulence exhibit pronounced breaks at certain wavenumbers which are related to the intrinsic scales of the media and are also strongly dependent on the actual degree of the wave nonlinearity points to possible use of the theory for extracting deep-ocean parameters from observed spectra of sea surface height. Really, SSH spectra based on satellite measurements contain information on the ocean stratification and deep-water processes in the thermocline because the baroclinic Rossby radius is a function of the density gradient while the degree of wave nonlinearity contains information about internal wave breaking, hence vertical mixing in the thermocline.

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References

Donelan, M. A., J. Hamilton, and W.H. Hui, 1985. Directional spectra of wind-generated waves. Phil. Trans. R. Sot. London, A 315,509-562.

Falkovich, G.E. and S.B. Medvedev. Kolmogorov-like spectrum for turbulence of inertial-gravity waves. Europhys. Let. 19(4), 279-284, 1992.

Forristall, Z., 1981, Measurements of a saturated range in ocean wave spectra. J. Geophys. Res., 86, 8075-8084

Frisch, U., Sulem, P.-L. & Nelkin, M., 1978. A simple dynamical model of intermittent fully developed turbulence. J. Fluid Mech., 87,719-736.

Fu, L.-L., 1983. On the wavenumber spectrum of oceanic mesoscale variability observed by the Seasat altimeter. J. Geophys. Res., 88(C7), 4331-4341.

Gaspar, P. and C. Wunsch, 1989. Estimates from altimeter data of baroclinic Rossby waves in the Northwestern Atlantic Ocean. Jour. Phys. Oceanogr., 19, 1821-1844.

Gill, A.E., 1982. Atmosphere-Ocean Dynamics. Academic press, New York, 662 pp.

Glazman, R.E., 1992, Multiwave interaction theory for wind-generated surface gravity waves. Journ. Fluid Mech., vol. 243, 623-635.

Glazman, R.E., 1993. A cascade model of wave turbulence with applications to surface gravity and capillary waves. Fractals, 1(3), 513-520.

Glazman, R.E., 1994. Surface gravity waves at equilibrium with a steady wind. J. Geophys. Res. 99(C3), 5249-5262.

Glazman, R.E., 1995(a). A simple theory of capillary-gravity wave turbulence. J. Fluid Mech., vol. 293, 25-34.

Glazman, R.E., 1995(b). Spectra of baroclinic inertia-gravity wave turbulence. Journ. Phys. Oceanogr. (Submitted for publication: Feb. 1995).

Gordon, A.L. and T.N. Baker. Ocean transients as observed by Geos-3 coincident orbits. J. Geophys. Res., 85(C1), 502-506, 1980.

Grose, P. L., K.L. Warsh and M. Garstang, 1972. Dispersion relations and wave shapes. J. Geophys. Res., 3902-3906

Hara, T., E.J. Bock, and D. Lyzenga, 1994. In situ measurements of capillary-gravity wave spectra using a scanning laser slope gauge and microwave radars. J. Geophys. Res., 99(C6), 12,593-12,602.

Hwang, P. A., Trizna, and J. Wu, Spatial measurements of short wind waves using a scanning slope sensor. Dyn. Atm. and Oceans 2.0, 1-23, 1993.

Kahma, K. K., 1981. A study of the growth of the wave spectrum with fetch. J. Phys. Oceanogr., 11, 1503-1515.

Kraichnan, R.H. Inertial ranges in two-dimensional turbulence. Phys. Fluids, 10(7), 1417-1423, 1967.

Jähne, B. and K.S. Riemer, 1990. Two-dimensional wave number spectra of small-scale water surface waves. Journ. Geophys. Res., 95(C7), 11,531-11,546.

Larraza, A., S.L. Garrett, and S. Putterman, 1990. Dispersion relations for gravity waves in a deep fluid: Second sound in a stormy sea. Phys. Rev. A 41(6), 3144-3155.

LeBlond, P.H. and L.A. Mysak, 1978, Waves In The Ocean. Elsivier, New York, 602 pp.

Le Traon, P.Y., M.C. Rouquet and Boissier, 1990. Spatial scales of mesoscale variability in the north Atlantic as deduced from Geosat data. J. Geophys. Res., 95(C1 1), 20,267-20,285.

Le Traon, P. Y., J. Stum, J. Dorandeu, and P. Gaspar, 1994. Global statistical analysis of Topex and Poseidon data. J. Geophys. Res., 99(C12), 24,619-24,631.

Phillips, O. M., 1958. The equilibrium range in the spectrum of wind-generated waves. J. Fluid Mech., 4, 426-434.

Phillips, O.M., 1977. The Dynamics of the Upper Ocean. Cambridge Univ. Press, New York, 336 pp.

Phillips, O. M., 1985. Spectral and statistical properties of the equilibrium range in wind-generated gravity waves. J. Fluid Mech., 156, 505-531.

Pierson,W.J. and L, Moskowitz, 1964. A proposed spectral model for fully developed wind seas based on the similarity theory of S.A. Kitaigorodskii. J. Geophys. Res., 69, 5181-5190.

Pierson,W.J., 1991. Comment. J. Geophys. Res., 96(C3),4973-4977.

Toba, Y., 1973. Local balance in the air-sea boundary processes, III, On the spectrum of wind waves. J. Oceanogr. Sot. Jpn, 29, 209-220.

Wunsch, C. and D. Stammer, 1995. The global frequency -wavenumber spectrum of oceanic variability estimated from Topex/Poseidon altimeter measurements. J. Geophys. Res. (Oceans). In press.

Zakharov, V. E., 1984. Kolmogorov spectra in weak turbulence problems. Handbook of Plasma Physics, Eds. M. N. Rosenbluth and R.Z. Sagdeev. Chapter 5.1. Elsevier Science Publishers, pp.4-36.

Zakharov, V.E. and. N.N. Filonenko, 1966, The energy spectrum for stochastic oscillation of a fluid's surface. Doklady Academii Nauk S. S. S. R., 170(6), 1292-1295, (in Russian)

Zakharov, V.E. and N.N. Filonenko, 1967. Weak turbulence of capillary waves. J. Appl. Math. Tech. Physics, No. 4,506-515.

Zakharov, V.E. and M.M. Zaslavskii, 1982. The kinetic equation and Kolmogorov spectra in the weak turbulence theory of wind waves. Izvestiya, Atmospheric and Oceanic Physics (English Translation), 18(9), 747-753

Zakharov, V. E., V.S. L'vov and G. Falkovich. 1992. Kolmogorov Spectra of Turbulence I: Wave Turbulence. Springer Verlag, Berlin, 264 pp.

Captions for Figures

Fig. 1. Theoretical and experimental spectra of "surface curvature" $B(k) = k^4 F_\eta(k)$ for wind speed values, U (m/see): 5.7, 7.0, 8.5 and 9.9- increasing upward.

(a) is based on equations (4.9),(4.10) and on an empirical formula for v as a function of wind: v(U) = 0.2 +0.6 U where U is in m/sec. The *energy* flux is determined as $Q = c\ U^3$ where the meaning and value of constant c are given in (Glazman, 1995(a)).

(b) is a subset of measurements reported by Hwang et al. (1993). (Reproduced by courtesy of the authors).

Fig. 2. 1-dimensional spectra of surface height variat ions observed by Topex altimeter along satellite "ground tracks" for mid-latitude regions. (Reproduced from (Le Traon et al., 1994) by courtesy of the authors).

Fig. 3. Non-dimensional spectra of surface height variations, $K\tilde{F}_\eta(K)$. Equation (4.30) is used for $K \leq 1.2$. Equation (4.27) is used for $K \geq 1.2$, for several values of the effective number of resonantly interacting Fourier harmonics, v, as designated at the curves.

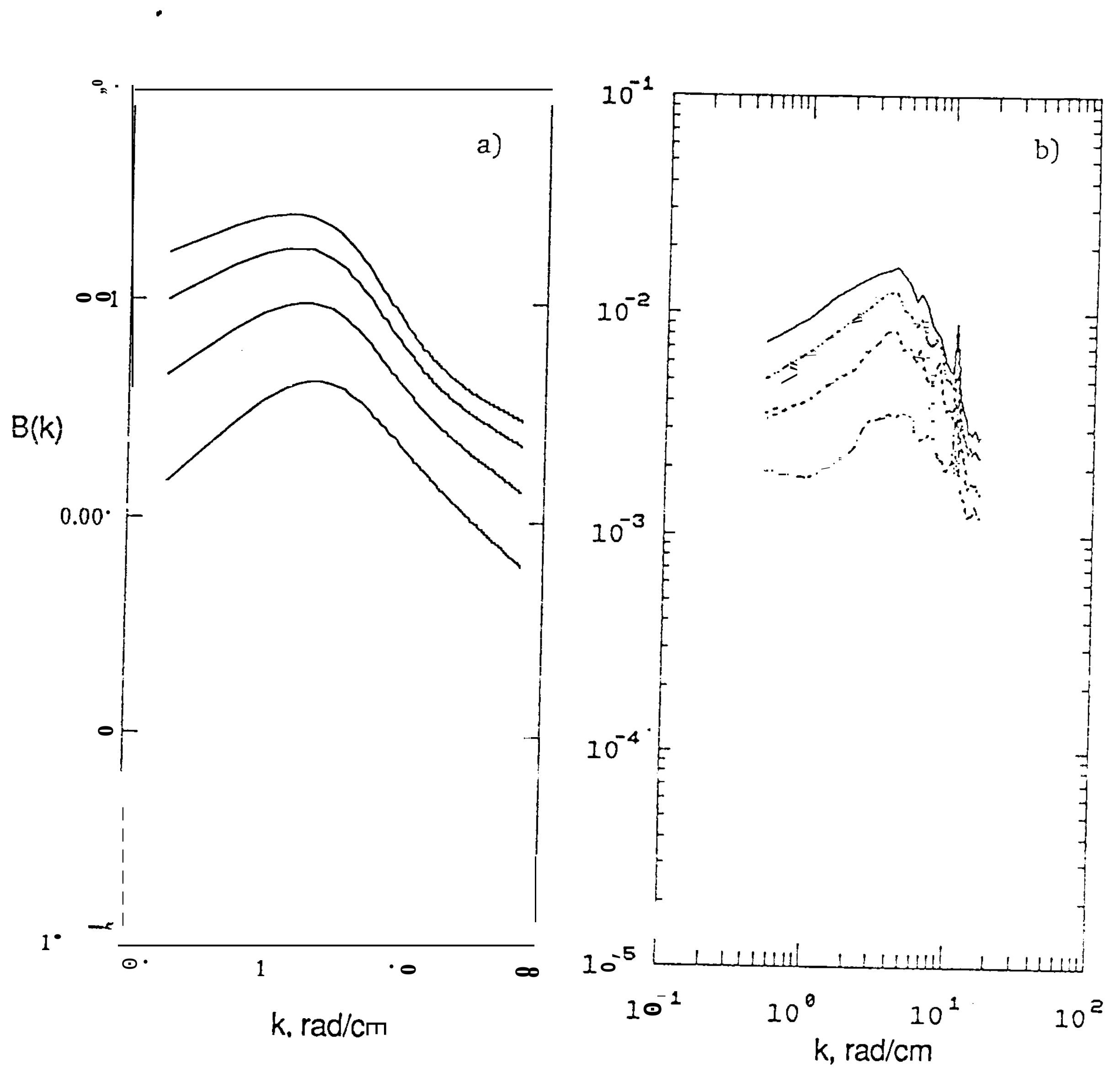
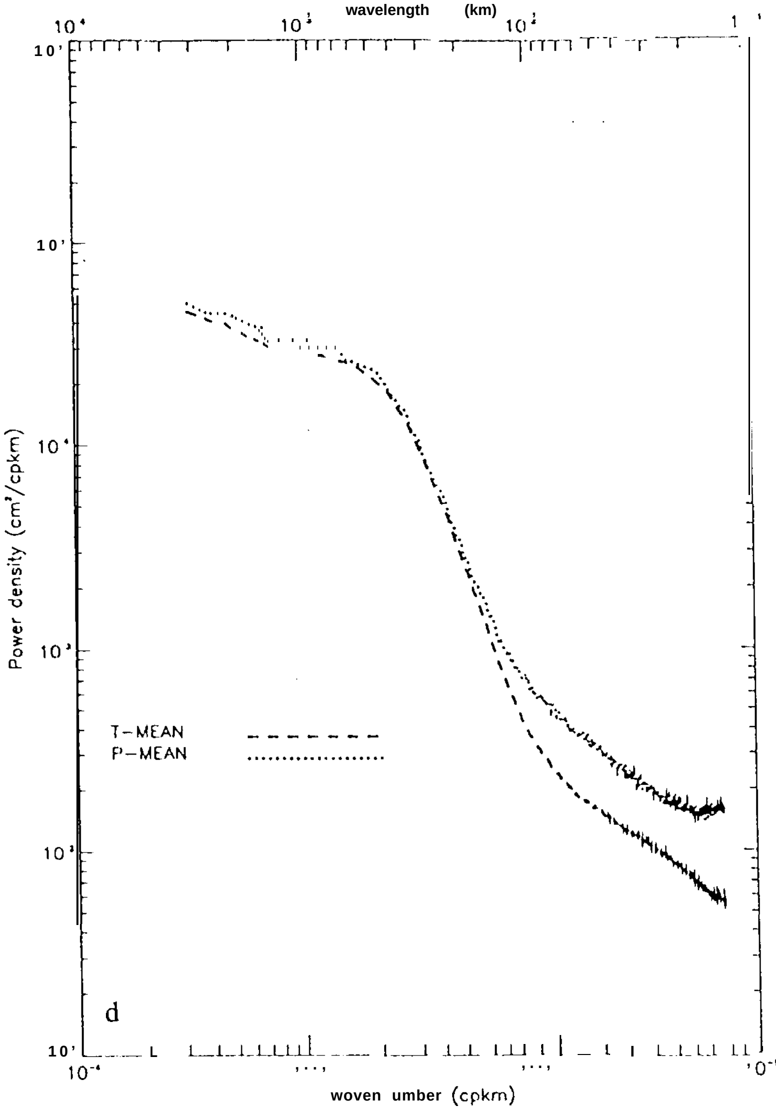


Fig.1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Power density (cm²/cpkm)
T-MEAN
P-MEAN
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woven umber (cpkm)

Fig. 2

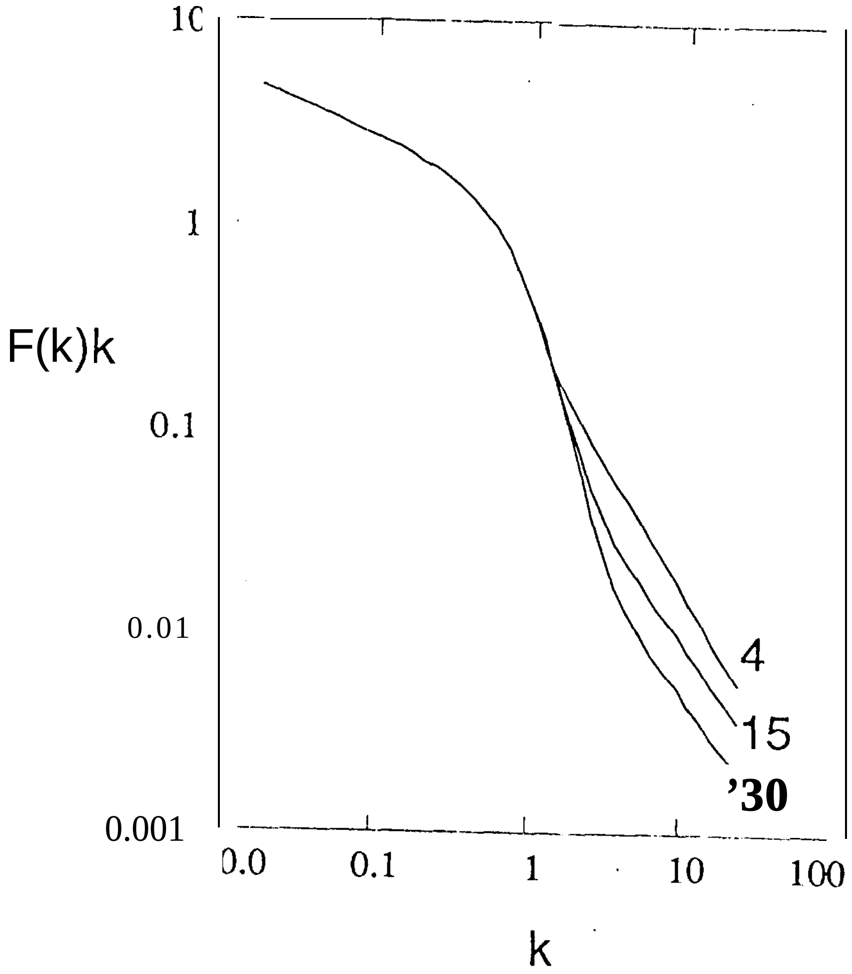

Fig. 3